

ON THE CHARACTER OF UNIPOTENT REPRESENTATIONS OF A SEMISIMPLE p -ADIC GROUP

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1. INTRODUCTION

1.1. Let K be a nonarchimedean local field and let \underline{K} be a maximal unramified field extension of K . Let \mathcal{O} (resp. $\underline{\mathcal{O}}$) be the ring of integers of K (resp. \underline{K}) and let \mathfrak{p} (resp. $\underline{\mathfrak{p}}$) be the maximal ideal of \mathcal{O} (resp. $\underline{\mathcal{O}}$). Let $\underline{K}^* = \underline{K} - \{0\}$.

Let G be a semisimple almost simple algebraic group defined and split over K with a given \mathcal{O} -structure compatible with the K -structure.

If V is an admissible representation of $G(K)$ of finite length, we denote by ϕ_V the character of V in the sense of Harish-Chandra, viewed as a \mathbf{C} -valued function on the set $G(K)_{rs} := G_{rs} \cap G(K)$. (Here G_{rs} is the set of regular semisimple elements of G and \mathbf{C} is the field of complex numbers.) In this paper we study the restriction of the function ϕ_V to a certain subset $G(K)_{cvr}$ of $G(K)_{rs}$, that is to the set of compact very regular elements in $G(K)$ (see 1.2), in the case where V is an irreducible unipotent representation of $G(K)$. (The notion of unipotent representation of $G(K)$ is defined in [L4] assuming that G is adjoint; the same definition can be given without this assumption.) We show (with some restriction on characteristic) that $\phi_V|_{G(K)_{cvr}}$ takes integer values which, at least if V is tempered and G is adjoint, are explicitly computable. (See Theorem 4.6). Our results apply in particular in the case where V is an irreducible admissible representation of $G(K)$ with nonzero vectors fixed by an Iwahori subgroup; they provide a partial generalization of the results in [KL] on the Steinberg representation of G (which are valid without restriction on characteristic).

1.2. Let $g \in G_{rs} \cap G(\underline{K})$. Let $T' = T'_g$ be the maximal torus of G that contains g . We say that g is very regular (resp. compact very regular) if T' is split over \underline{K} and for any root α with respect to T' viewed as a homomorphism $T'(\underline{K}) \rightarrow \underline{K}^*$ we have

$$\alpha(g) \notin (1 + \underline{\mathfrak{p}}) \text{ (resp. } \alpha(g) \in \underline{\mathcal{O}}, \alpha(g) \notin (1 + \underline{\mathfrak{p}})).$$

Let $G(\underline{K})_{vr}$ (resp. $G(\underline{K})_{cvr}$) be the set of elements in $G(\underline{K})$ which are very regular (resp. compact very regular). Note that $G(\underline{K})_{cvr} \subset G(\underline{K})_{vr} \subset G_{rs}$. We write $G(K)_{vr} = G(\underline{K})_{vr} \cap G(K)$, $G(K)_{cvr} = G(\underline{K})_{cvr} \cap G(K)$.

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1.3. Notation. Let \underline{K}_* be the set of all $x \in \underline{K}$ such that $x^n = 1$ for some $n \geq 1$ prime to p . We write $\mathcal{O}/\mathfrak{p} = F_q$, a finite field with q elements, of characteristic p .

We fix a maximal torus T of G defined and split over K . Let X be the group of characters of T . Let $R \subset X$ be the set of roots with respect to T . For each $\alpha \in R$ let U_α be the corresponding root subgroup of G .

Let \mathcal{P} (resp. $\underline{\mathcal{P}}$) be the set of parahoric subgroups of $G(K)$ (resp. of $G(\underline{K})$). Let $G(K)'$ (resp. $G(\underline{K})'$) be the derived subgroup of $G(K)$ (resp. $G(\underline{K})$).

2. COMPACT VERY REGULAR ELEMENTS

2.1. Let $F : \underline{K} \rightarrow \underline{K}$ be the Frobenius automorphism, that is the field automorphism whose restriction to \underline{K}_* is the map $x \mapsto x^q$; then K is the fixed point set of F . Now F induces a group isomorphism $G(\underline{K}) \rightarrow G(\underline{K})$ (denoted again by F) whose fixed point set is $G(K)$. Note that F maps any parahoric subgroup of $G(\underline{K})$ onto a parahoric subgroup of $G(\underline{K})$. The map $\underline{P} \mapsto \underline{P} \cap G(K)$ is a 1 – 1 correspondence between the set $\{\underline{P} \in \underline{\mathcal{P}}; F(\underline{P}) = \underline{P}\}$ and \mathcal{P} .

Let $\underline{\mathcal{P}}_\emptyset$ be the set of $\underline{P} \in \underline{\mathcal{P}}$ that are Iwahori subgroups. The minimal elements (under inclusion) of $\underline{\mathcal{P}} - \underline{\mathcal{P}}_\emptyset$ fall into finitely many $G(\underline{K})'$ -orbits $(\underline{\mathcal{P}}_i)_{i \in \mathbf{I}}$ (under conjugation); here \mathbf{I} is a finite indexing set. If $J \subsetneq \mathbf{I}$ let $\underline{\mathcal{P}}_J$ be the set of $\underline{P} \in \underline{\mathcal{P}}$ such that the following holds: for $i \in \mathbf{I}$, \underline{P} contains a subgroup in $\underline{\mathcal{P}}_i$ if and only if $i \in J$. The sets $\underline{\mathcal{P}}_J$ with $J \subsetneq \mathbf{I}$ are exactly the $G(\underline{K})'$ -orbits on the set of parahoric subgroups of $G(\underline{K})$. This agrees with the earlier notation $\underline{\mathcal{P}}_\emptyset$; for $i \in \mathbf{I}$ we have $\underline{\mathcal{P}}_{\{i\}} = \underline{\mathcal{P}}_i$.

Let T' be a maximal torus of G which is defined and split over \underline{K} . We define two subgroups $T'_*, T'_!$ of $T'(\underline{K})$ as follows. Let Y' be the group of cocharacters of T' . We can identify $\underline{K} \otimes Y' = T'(\underline{K})$ in an obvious way. Let T'_* (resp. $T'_!$) be the subgroup of $T'(\underline{K})$ corresponding to $\underline{K}_* \otimes Y$ (resp. $(1 + \mathfrak{p}) \otimes Y$) under this isomorphism; thus T'_* is the set of all $t \in T'(\underline{K})$ such that $t^n = 1$ for some $n \geq 1$ prime to p . Let $\underline{\mathcal{P}}_\emptyset^{T'_*} = \{\underline{B}' \in \underline{\mathcal{P}}_\emptyset; T'_* \subset \underline{B}'\}$; this set is nonempty.

We now fix $\underline{B} \in \underline{\mathcal{P}}_\emptyset^{T'_*}$ (T as in 1.3).

Let W be an indexing set for the set of $(\underline{B}, \underline{B})$ -double cosets in $G(\underline{K})$. We denote by O_w the double coset corresponding to $w \in W$. Let $W' = \{w \in W; O_w \subset G(\underline{K})'\}$ and let $\Omega = \{w \in W; O_w \subset N(\underline{B})\}$; here $N(\underline{B})$ is the normalizer of \underline{B} in $G(\underline{K})$. If $i \in \mathbf{I}$ let \underline{P}_i be the unique element of $\underline{\mathcal{P}}_i$ containing \underline{B} . Then $\underline{P}_i - \underline{B} = O_w$ for a well defined element $w \in W'$; we set $w = s_i$. For $J \subsetneq \mathbf{I}$ let $\underline{\mathcal{P}}_J$ be the unique subgroup in $\underline{\mathcal{P}}_J$ that contains \underline{B} . Let W_J be the subgroup of W' generated by $\{s_i; i \in J\}$.

Two $(\underline{B}, \underline{B})$ -double cosets $O_w, O_{w'}$ are said to be composable if there exists $w'' \in W$ such that multiplication defines a bijection $O_w \times_{\underline{B}} O_{w'} \rightarrow O_{w''}$; here \underline{B} acts on $O_w \times O_{w'}$ by $b : (g, g') \mapsto (gb^{-1}, bg')$. We then set $w \cdot w' = w''$. There is a unique group structure on W such that the product of w, w' is $w \cdot w'$ whenever $O_w, O_{w'}$ are composable and $s_i^2 = 1$ for any $i \in \mathbf{I}$. (The unit element is the $(\underline{B}, \underline{B})$ -double coset \underline{B} .) Then W' is the subgroup of W generated by $\{s_i; i \in \mathbf{I}\}$.

This is a normal subgroup of W and Ω is an abelian subgroup of W which maps isomorphically onto W/W' . The group W' together with $\{s_i; i \in \mathbf{I}\}$ is a Coxeter group (in fact, an affine Weyl group) with length function $l : W' \rightarrow \mathbf{N}$. We extend this to a function $l : W \rightarrow \mathbf{N}$ by $l(w_1 w_2) = l(w_2 w_1) = l(w_1)$ for $w_1 \in W', w_2 \in \Omega$. Then $O_w, O_{w'}$ are composable precisely when $l(w w') = l(w) + l(w')$.

Now any $\underline{P} \in \mathcal{P}$ has a prounipotent radical $U_{\underline{P}}$ (a normal subgroup) such that $\underline{P}/U_{\underline{P}}$ is naturally a connected reductive group over $\underline{\mathcal{O}}/\underline{\mathfrak{p}}$ (an algebraic closure of F_q).

The orbits of the diagonal conjugation action of $G(\underline{K})'$ on $\underline{\mathcal{P}}_{\emptyset} \times \underline{\mathcal{P}}_{\emptyset}$ are indexed by the elements of W' : to $w \in W'$ corresponds the $G(\underline{K})'$ -orbit of $(\underline{B}, g \underline{B} g^{-1})$ where g is some/any element of O_w . We write $\text{pos}(\underline{B}', \underline{B}'') = w$ when the $G(\underline{K})'$ -orbit of $(\underline{B}', \underline{B}'') \in \underline{\mathcal{P}}_{\emptyset} \times \underline{\mathcal{P}}_{\emptyset}$ corresponds to $w \in W'$.

For any $w \in W'$ there is a unique subgroup $\underline{B}_w \in \underline{\mathcal{P}}_{\emptyset}^{T*}$ such that $\text{pos}(\underline{B}, \underline{B}_w) = w$. (We have $\underline{B}_1 = \underline{B}$.) Moreover we have

$$(a) \quad \underline{\mathcal{P}}_{\emptyset}^{T*} = \{\underline{B}_w; w \in W'\}.$$

Let $\underline{\mathcal{P}}_{\emptyset}^{T*!}$ be the set of all $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}$ such that \underline{B}' contains the subgroup $T_{*!} := T_* T_!$ of $T(K)$. We have $T_{*!} \subset \underline{B}_w$ for any $w \in W'$ hence $\underline{\mathcal{P}}_{\emptyset}^{T*} = \underline{\mathcal{P}}_{\emptyset}^{T*!}$.

2.2. Let $g \in G(\underline{K})_{crr}$. Let $T' = T'_g$. Recall that T' is defined and split over \underline{K} . We have $g \in T'_{*!}$ (see 2.1) hence $g = g_* g_!$ where $g_* \in T'_*$, $g_! \in T'_!$. From the definitions we see that $g_* \in G(\underline{K})_{crr}$. Let $\underline{\mathcal{P}}_{\emptyset}^g$ (resp. $\underline{\mathcal{P}}_{\emptyset}^{g*}$) be the set of $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}$ such that $g \in \underline{B}'$ (resp. $g_* \in \underline{B}'$). We show:

$$(a) \quad \underline{\mathcal{P}}_{\emptyset}^g = \underline{\mathcal{P}}_{\emptyset}^{g*} = \underline{\mathcal{P}}_{\emptyset}^{T'*} = \underline{\mathcal{P}}_{\emptyset}^{T*!}.$$

(b) Given $w \in W'$ and $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}^{T'*}$ (resp. $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}^g$) there is a unique subgroup $\underline{B}'' \in \underline{\mathcal{P}}_{\emptyset}^{T'*}$ (resp. $\underline{B}'' \in \underline{\mathcal{P}}_{\emptyset}^g$) such that $\text{pos}(\underline{B}', \underline{B}'') = w$. We set $\underline{B}'' = w \circ \underline{B}'$. Now $w : \underline{B}' \mapsto w \circ \underline{B}'$ defines a free transitive action of W' on the set $\underline{\mathcal{P}}_{\emptyset}^{T'*}$ (resp. $\underline{\mathcal{P}}_{\emptyset}^g$).

We prove (a). We can assume that $T' = T$. The third equality in (a) is known from 2.1. The inclusions $\underline{\mathcal{P}}_{\emptyset}^{T*!} \subset \underline{\mathcal{P}}_{\emptyset}^g$, $\underline{\mathcal{P}}_{\emptyset}^{T*} \subset \underline{\mathcal{P}}_{\emptyset}^{g*}$ are obvious. We prove the inclusion $\underline{\mathcal{P}}_{\emptyset}^{g*} \subset \underline{\mathcal{P}}_{\emptyset}^{T*}$. For any $w \in W'$ there exists a subgroup Z_w of $G(\underline{K})$ normalized by T_* and consisting of unipotent elements in G such that the map $\psi : \sqcup_{w \in W'} Z_w \rightarrow \underline{\mathcal{P}}_{\emptyset}$ given by $h \mapsto h \underline{B}_w h^{-1}$ (where $h \in Z_w$) is a bijection. (Assume for simplicity that we are in the equal characteristic case. Let I be an "anti-Iwahori" subgroup of $G(\underline{K})$ such that $I \cap \underline{B} = T_*$. Then we can take $Z_w = U_{\underline{B}} \cap \dot{w} I \dot{w}^{-1}$ where $\dot{w} \in O_w$ normalizes T_* . The fact that Z_w consists of unipotent elements is seen as in [L5, 4.2].) Now ψ is T_* -equivariant where T_* acts on each Z_w and on $\underline{\mathcal{P}}_{\emptyset}$ by conjugation. Let $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}$ be such that $g_* \in \underline{B}'$. We have $\psi^{-1}(\underline{B}') = z \in Z_w$ for some $w \in W'$. We have $g_* \underline{B}' g_*^{-1} = \underline{B}'$ hence $g_* z g_*^{-1} = z$ since $g_* \in T_*$ (we use the T_* -equivariance of ψ). Since $g_* \in G_{rs}$ and z is unipotent, the equality $g_* z g_*^{-1} = z$ implies that $z = 1$. Hence $\underline{B}' = \underline{B}_w$. Thus $\underline{\mathcal{P}}_{\emptyset}^{g*} \subset \underline{\mathcal{P}}_{\emptyset}^{T*}$, as claimed. To prove (a), it remains to show that $\underline{\mathcal{P}}_{\emptyset}^g \subset \underline{\mathcal{P}}_{\emptyset}^{T*}$. The previous argument shows that for any $w \in W'$, $\text{Ad}(g_*)$ has a unique fixed point on

the space $\{\underline{B}'' \in \underline{\mathcal{P}}_\emptyset; \text{pos}(\underline{B}, \underline{B}'') = w\}$, namely \underline{B}_w . Now $\text{Ad}(g)$ acts on the same space as the composition of two commuting maps $\text{Ad}(g_*)$ and $\text{Ad}(g!)$ the first of which has finite order prime to p while the second has finite order equal to a power of p . Thus, a fixed point of the composition must be also a fixed point of $\text{Ad}(g_*)$ hence it must be \underline{B}_w . Thus, $\underline{\mathcal{P}}_\emptyset^g \subset \underline{\mathcal{P}}_\emptyset^{T^*}$, as claimed. This proves (a).

We prove (b). We can assume that $T' = T$. In this case the result follows from (a) and 2.1(a).

2.3. Let $\widetilde{G(\underline{K})_{cvt}}$ be the set of all pairs (g, \underline{B}') where $g \in G(\underline{K})_{cvt}$ and $\underline{B}' \in \underline{\mathcal{P}}_\emptyset^g$. By 2.2(b), $w : (g, \underline{B}') \mapsto (g, w \circ \underline{B}')$ is a free action of W' on $\widetilde{G(\underline{K})_{cvt}}$ whose orbits are exactly the fibres of the first projection $pr_1 : \widetilde{G(\underline{K})_{cvt}} \rightarrow G(\underline{K})_{cvt}$.

Now let A be a (finite dimensional) representation of W' . For any $g \in G(\underline{K})_{cvt}$ we define a vector space A_g as the set of the W' -orbits on the set $\underline{\mathcal{P}}_\emptyset^g \times A$ for the free W' -action $w : (\underline{B}', a) \mapsto (w \circ \underline{B}', wa)$. We can think of the union $\sqcup_{g \in G(\underline{K})_{cvt}} A_g$ and its obvious projection to $G(\underline{K})_{cvt}$ as something like a "local system" \mathcal{L}_A on $G(\underline{K})_{cvt}$ associate to A and the principal W' -covering pr_1 above.

We now define an isomorphism of local systems $\psi : F^* \mathcal{L}_A \xrightarrow{\sim} \mathcal{L}_A$ as the collection of isomorphisms $\psi_g : A_{F(g)} \rightarrow A_g$ (with $g \in G(\underline{K})_{cvt}$) where ψ_g is induced by

$$\underline{\mathcal{P}}_\emptyset^{F(g)} \times A \rightarrow \underline{\mathcal{P}}_\emptyset^g \times A, (\underline{B}', a) \mapsto (F^{-1}(\underline{B}'), a)$$

by passage to W' -orbits.

We define $\tau_A : G(\underline{K})_{cvt} \rightarrow \mathbf{C}$ by $\tau_A(g) = \text{tr}(\psi_g : A_g \rightarrow A_g)$.

2.4. Let $g \in G(\underline{K})_{cvt}$. Let $T' = T'_g$ and let g_* be as in 2.2. Let $\underline{P} \in \underline{\mathcal{P}}$.

(a) The following four conditions are equivalent: (i) $g \in \underline{P}$; (ii) $g_* \in \underline{P}$; (iii) $T'_* \subset \underline{P}$; (iv) $T'_{*!} \subset \underline{P}$.

Assume that $g \in \underline{P}$. We can find a Borel subgroup β of $\underline{P}/U_{\underline{P}}$ which contains the image of \bar{g} of g in $\underline{P}/U_{\underline{P}}$. The inverse image of β under $\underline{P} \rightarrow \underline{P}/U_{\underline{P}}$ is an Iwahori subgroup \underline{B}' of $G(\underline{K})$ which contains g and is contained in \underline{P} . By 2.2(a), we have $T'_{*!} \subset \underline{B}'$ hence $T'_{*!} \subset \underline{P}$. Thus (i) implies (iv). An entirely similar proof shows that (ii) implies (iv). It is obvious that (iv) implies (iii), that (iii) implies (ii) and that (iv) implies (i). This proves (a).

We show:

(b) If $g \in \underline{P}$, \bar{g} are as in (a) then \bar{g} is regular semisimple in $\underline{P}/U_{\underline{P}}$.

The image of T'_* under $\underline{P} \rightarrow \underline{P}/U_{\underline{P}}$ is a maximal torus \bar{T}' of $\underline{P}/U_{\underline{P}}$ that contains \bar{g} . For any root α of $\underline{P}/U_{\underline{P}}$, viewed as a character $\bar{T}' \rightarrow \mathcal{O}/\mathfrak{p} - \{0\}$ the value $\alpha(\bar{g})$ is equal to $\alpha'(g_*) \in \underline{K}_*$ for some root α' of G viewed as a character of T' . (We identify $\underline{K}_* = \mathcal{O}/\mathfrak{p} - \{0\}$ in an obvious way.) Since $\alpha'(g_*) \neq 1$ we must have $\alpha(\bar{g}) \neq 1$. This proves (b).

2.5. Let T' be a maximal torus of G which is defined over K and is \underline{K} -split. We show:

(a) There is a unique W' -conjugacy class $C_{T'}$ in W' such that the following holds: for any $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}^{T'}$ we have $\text{pos}(\underline{B}', F(\underline{B}')) = w$ for some $w \in C_{T'}$.

It is enough to show that if $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}^{T'}$, $\underline{B}'' \in \underline{\mathcal{P}}_{\emptyset}^{T'}$ and w', w'' in W' are given by $\text{pos}(\underline{B}', F(\underline{B}')) = w'$, $\text{pos}(\underline{B}'', F(\underline{B}'')) = w''$ then $w'' = yw'y^{-1}$ for some $y \in W'$. We have $F(\underline{B}') \in \underline{\mathcal{P}}_{\emptyset}^{T'}$, $F(\underline{B}'') \in \underline{\mathcal{P}}_{\emptyset}^{T'}$. From the definitions we have $F(\underline{B}') = w' \circ \underline{B}'$, $F(\underline{B}'') = w'' \circ \underline{B}''$. We have $\underline{B}'' = y \circ \underline{B}'$ for a unique $y \in W'$. It follows that $F(\underline{B}'') = y \circ F(\underline{B}')$. (Note that F induces the identity map on W' .) Thus $w'' \circ (y \circ \underline{B}') = y \circ (w' \circ \underline{B}')$ that is $(w''y) \circ \underline{B}' = (yw') \circ \underline{B}'$. By the freeness of the W' -action on $\underline{\mathcal{P}}_{\emptyset}^{T'}$ we then have $w''y = yw'$. This proves (a).

We show:

(b) Any element of $C_{T'}$ has finite order in W' .

Let $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}^{T'}$. We set $\text{pos}(\underline{B}', F(\underline{B}')) = w$ so that $w \in C_{T'}$. Now T' becomes split after a finite unramified extension of K . Hence there exists $s \geq 1$ such that $F^s(\underline{B}') = \underline{B}'$. We have $F(\underline{B}') = w \circ \underline{B}'$, $F^2(\underline{B}') = w \circ (F(\underline{B}')) = w^2 \circ \underline{B}'$, \dots , $F^s(\underline{B}') = w^s \circ \underline{B}'$. Thus $w^s \circ \underline{B}' = \underline{B}'$ so that $w^s = 1$. This proves (b).

Now let A, τ_A be as in 2.3. From the definitions we see that for $g \in G(K)_{\text{cvt}}$ we have

$$(c) \tau_A(g) = \text{tr}(w^{-1}, A) = \text{tr}(w, A)$$

where $T' = T'_g$, $w \in C_{T'}$. In particular, we have $\tau_A(g) \in \mathbf{Z}$. (Indeed we can assume that $w \in W_H$ for some $H \subsetneq \mathbf{I}$ so that $\text{tr}(w, A) = \text{tr}(w, A|_{W_H})$ and we use the rationality of the irreducible representations of W_H .)

2.6. In [Ge, 3.3], Gérardin defines a map

$$\mu : \{G(K)' - \text{conjugacy classes of } \underline{K} - \text{split maximal tori in } G \text{ defined over } K\} \rightarrow \{W' - \text{conjugacy classes of elements of finite order in } W'\}$$

and shows that

(a) μ is a bijection.

(Actually in [Ge] it is assumed that G is simply connected so that $G(K)' = G(K)$ but the general case can be deduced from this.) One can show that μ coincides with the map induced by $T' \mapsto C_{T'}$ in 2.5.

2.7. Let $g \in G(K)_{\text{cvt}}$. Let $T' = T'_g$ and let $C = C_{T'}$. Using 2.5(b) we see that we can find $H \subsetneq \mathbf{I}$ and $w \in W_H$ such that $w \in C$; moreover we can assume that w is elliptic in W_H . We show:

(a) There exists $\underline{P} \in \underline{\mathcal{P}}_H$ such that $F(\underline{P}) = \underline{P}$ and $T'_* \subset \underline{P}$ (hence $g \in \underline{P}$, see 2.4(a)).

Let $\underline{B}'' \in \underline{\mathcal{P}}_{\emptyset}^{T'}$. Then $F(\underline{B}'') = w' \circ \underline{B}''$ for a unique $w' \in C$. We have $w' = ywy^{-1}$ for some $y \in W'$. Setting $\underline{B}' = y^{-1} \circ \underline{B}''$ we have $\underline{B}' \in \underline{\mathcal{P}}_{\emptyset}^{T'}$, $F(\underline{B}') = w \circ \underline{B}'$ hence $\text{pos}(\underline{B}', F(\underline{B}')) = w$. Define $\underline{P} \in \underline{\mathcal{P}}_H$ by $\underline{B}' \subset \underline{P}$. Then we have automatically $F(\underline{B}') \subset \underline{P}$ (since $\text{pos}(\underline{B}', F(\underline{B}')) \in W_H$). We have also $F(\underline{B}') \subset F(\underline{P})$. Since

$F(\underline{B}')$ is contained in a unique parahoric subgroup in $\underline{\mathcal{P}}_H$ we must have $F(\underline{P}) = \underline{P}$. We have $T'_* \subset \underline{P}$ and (a) is proved.

Using the obvious identification $\underline{K}_* = \underline{\mathcal{O}}/\underline{\mathfrak{p}} - \{0\}$ we can view T'_* as the torus over $\underline{\mathcal{O}}/\underline{\mathfrak{p}}$ (identified with its group of $\underline{\mathcal{O}}/\underline{\mathfrak{p}}$ points). This torus has a natural F_q -structure (induced by F) hence it has a maximal F_q -split subtorus T'_{*s} (it is the identity component of $\{t \in T'_*; t^q = t\}$). Let \underline{P} be as in (a). The image of T'_{*s} under $pr : \underline{P} \rightarrow \underline{P}/U_{\underline{P}}$ is a torus of $\underline{P}/U_{\underline{P}}$ defined and split over F_q and we can find a Borel subgroup of $\underline{P}/U_{\underline{P}}$ defined over F_q that contains this torus. The inverse image of this Borel subgroup under pr is an Iwahori subgroup \underline{B}' of $G(K)$ such that $\underline{B}' \subset \underline{P}$, $F(\underline{B}') = \underline{B}'$ and $T'_{*s} \subset \underline{B}'$. Now we can find a maximal torus T'' of G defined and split over K such that $T''_* \subset \underline{B}'$. Now T'' viewed as a torus over $\underline{\mathcal{O}}/\underline{\mathfrak{p}}$ is an F_q -split torus in \underline{B}' . We can find $x \in \underline{B}' \cap G(K)$ such that $T'_{*s} \subset xT''_*x^{-1}$. Replacing T'' by $xT''x^{-1}$ we can assume that $T'_{*s} \subset T''_* \subset \underline{B}'$. We can find $z \in G(K)'$ such that $zT''z^{-1} = T$, $z\underline{B}'z^{-1} = \underline{B}$. Let $g' = zgz^{-1}$, $\underline{P}' = z\underline{P}z^{-1}$. We have $\underline{B} \subset \underline{P}'$, $\underline{P}' \in \underline{\mathcal{P}}_H$ hence $\underline{P}' = \underline{P}_H$. We have $T'_{g'} = zT'z^{-1}$, $(zT'z^{-1})_{*s} \subset T''_*$, $g' \in \underline{P}'$. Thus we have the following result.

(b) We can find a $G(K)'$ -conjugate g' of g such that $g' \in \underline{P}_H$, $(T'_{g'})_{*s} \subset T''_*$.

2.8. Assume that $g \in G(K)_{cwr}$, $g' \in G(K)_{cwr}$ are contained in the same maximal torus T' . Let $\underline{P} \in \underline{\mathcal{P}}$. The following two conditions are equivalent: (i) $g \in \underline{P}$ and (ii) $g' \in \underline{P}$. Indeed, by 2.4(a) both conditions are equivalent to the condition that $T'_* \subset \underline{P}$. It follows that:

(a) If \underline{P} is a minimal F -stable parahoric subgroup of $G(\underline{K})$ containing g then \underline{P} is a minimal F -stable parahoric subgroup of $G(\underline{K})$ containing g' .

3. RECOLLECTIONS ON AFFINE HECKE ALGEBRAS

3.1. We now review some definitions and results of [L4]. (In [L4], G is assumed to be adjoint but the results of [L4] that we use in this section extend with the same proof to the general case.)

We set $B := \underline{B} \cap G(K)$; this is an Iwahori subgroup of $G(K)$.

For any $J \subsetneq \mathbf{I}$ let \mathcal{P}_J be the set of $P \in \mathcal{P}$ of the form $\underline{P} \cap G(K)$ where $\underline{P} \in \underline{\mathcal{P}}_J$ is F -stable. The sets \mathcal{P}_J are exactly the orbits of $G(K)'$ on \mathcal{P} (under conjugation). Let P_J be the unique subgroup in \mathcal{P}_J that contains B .

Now any $P \in \mathcal{P}$ has a pronipotent radical U_P (a normal subgroup) such that the quotient $\bar{P} = P/U_P$ is naturally the group of F_q -points of a connected reductive group defined and split over F_q .

3.2. We now fix $J \subsetneq \mathbf{I}$ and a unipotent cuspidal (irreducible) representation (over \mathbf{C}) of the finite group \bar{P} where $P = P_J$; let \mathbf{E} be the vector space (over \mathbf{C}) of this representation. We regard \mathbf{E} as a P -module via the surjective homomorphism $P \rightarrow \bar{P}$.

Let \mathcal{X} be the \mathbf{C} -vector space consisting of all functions $f : G(K) \rightarrow \mathbf{E}$ such that $f(gh) = h^{-1}f(g)$ for all $g \in G(K)$, $h \in P$ and $\text{supp}(f) = \{g \in G(K); f(g) \neq 0\}$ is contained in a union of finitely many P -cosets in $G(K)/P$. For $g' \in G(K)$, $f \in$

\mathcal{X} we define $g'f : G(K) \rightarrow \mathbf{E}$ by $(g'f)(g) = f(g'^{-1}g)$. Then $g'f \in \mathcal{X}$. This defines a representation of $G(K)$ on the vector space \mathcal{X} . We have a direct sum decomposition $\mathcal{X} = \bigoplus_{gP} \mathcal{X}_{gP}$ where gP runs over the cosets $G(K)/P$ and $\mathcal{X}_{gP} = \{f \in \mathcal{X}; \text{supp}(f) \in gP\}$. Let \mathcal{H} be the endomorphism algebra of the $G(K)$ -module \mathcal{X} . For any (P, P) -double coset Θ in $G(K)$ we denote by \mathcal{H}_Θ the set consisting of those $\psi \in \mathcal{H}$ such that $\psi(\mathcal{X}_{gP}) \subset \bigoplus_{g'P \in G(K)/P; g'^{-1}g \in \Theta} \mathcal{X}_{g'P}$ for some $gP \in G(K)/P$ or equivalently any $gP \in G(K)/\mathbf{P}$. We say that Θ (as above) is good if for some (or equivalently any) $g \in \Theta$ we have $U_P(P \cap gPg^{-1}) = P$. If Θ is a good double coset then $\dim \mathcal{H}_\Theta = 1$; if Θ is not a good double coset then $\mathcal{H}_\Theta = 0$. Hence $\mathcal{H} = \bigoplus_{\Theta} \mathcal{H}_\Theta$ where Θ runs over the good (P, P) -double cosets in $G(K)$.

3.3. Let $\text{Irr}(G(K); P, \mathbf{E})$ be the set of irreducible admissible representations (V, σ) of $G(K)$ (up to isomorphism) with the following property: the (necessarily finite dimensional) vector space V^{U_P} of U_P -invariants, regarded as a \bar{P} -module in an obvious way, contains the \bar{P} -module \mathbf{E} . An irreducible admissible representation of $G(K)$ is said to be unipotent if it belongs to $\text{Irr}(G(K); P, \mathbf{E})$ for some P, \mathbf{E} as in 3.2.

We now fix again P, \mathbf{E} as in 3.2 and we fix $(V, \sigma) \in \text{Irr}(G(K); P, \mathbf{E})$. By a version of Frobenius reciprocity we have an isomorphism $\text{Hom}_{G(K)}(\mathcal{X}, V) \xrightarrow{\sim} \text{Hom}_P(\mathbf{E}, V)$ induced by the imbedding $\mathbf{E} \rightarrow \mathcal{X}$ which takes $e \in \mathbf{E}$ to the function $f_e : G(K) \rightarrow \mathbf{E}$ given by $h \mapsto h^{-1}e$ if $h \in P$ and $g \mapsto 0$ if $g \notin P$. Since $\text{Hom}_{G(K)}(\mathcal{X}, V)$ is an \mathcal{H} -module in an obvious way, we see that $\text{Hom}_P(\mathbf{E}, V)$ is naturally an \mathcal{H} -module. Clearly we have $\text{Hom}_P(\mathbf{E}, V) = \text{Hom}_{\bar{P}}(\mathbf{E}, V^{U_P})$. Hence $\text{Hom}_{\bar{P}}(\mathbf{E}, V^{U_P})$ (a finite dimensional \mathbf{C} -vector space) becomes an \mathcal{H} -module.

If Θ is a (P, P) -double coset in $G(K)$ then Θ is a union $\bigcup_w (O_w \cap G(K))$ where w runs over a (W_J, W_J) -double coset in W . This gives a 1–1 correspondence between the set of (P, P) -double cosets in $G(K)$ and the set of (W_J, W_J) -double cosets in W . A (P, P) -double coset is good if and only if the corresponding (W_J, W_J) -double coset in W is contained in NW_J , the normalizer of W_J in W . In each (W_J, W_J) -double coset in W contained in NW_J there is a unique element of minimal length. Hence the set \mathcal{W} consisting of the elements of minimal length in the various (W_J, W_J) -double cosets in W contained in NW_J is an indexing set for the set of good (P, P) -double cosets in $G(K)$. We denote the good (P, P) -double coset corresponding to an element $w \in \mathcal{W}$ by Θ_w . Actually, \mathcal{W} is a subgroup of NW_J . We have $\mathcal{H} = \bigoplus_{w \in \mathcal{W}} \mathcal{H}_{\Theta_w}$ with $\dim \mathcal{H}_{\Theta_w} = 1$ for any $w \in \mathcal{W}$. As in [L4] we see that this decomposition gives rise to a description of \mathcal{H} as an (extended) affine Hecke algebra with explicitly known (possibly unequal) parameters.

3.4. Now let $H \subsetneq \mathbf{I}$ be such that $J \subset H$ and let $Q = P_H$, $\underline{Q} = \underline{P}_H$. Now $\beta := B/U_Q$ is the group of F_q -points of a Borel subgroup defined over F_q of the reductive group $\underline{Q}/U_{\underline{Q}}$ defined and split over F_q and $\pi := P/U_Q$ is the group of F_q -points of a parabolic subgroup of that reductive group with $\beta \subset \pi$. Let $\kappa : Q \rightarrow \bar{Q}$ be the canonical homomorphism. The subsets $\kappa(O_w \cap G(K))$ ($w \in W_H$) are exactly

the (β, β) -double cosets in \bar{Q} hence W_H can be identified with the Weyl group of Q/U_Q . For any parabolic subgroup π' of \bar{Q} we denote by $u_{\pi'}$ the unipotent radical of π' .

Let \mathcal{X}_H be the \mathbf{C} -vector space consisting of all functions $f : \bar{Q} \rightarrow \mathbf{E}$ such that $f(gh) = h^{-1}f(g)$ for all $g \in \bar{Q}, h \in \pi$. (Note that \mathbf{E} is naturally a π -module on which $u_\pi := U_P/U_Q$ acts trivially.)

For $g' \in \bar{Q}, f \in \mathcal{X}_H$ we define $g'f : \bar{Q} \rightarrow \mathbf{E}$ by $(g'f)(g) = f(g'^{-1}g)$. Then $g'f \in \mathcal{X}_H$. This defines a representation of \bar{Q} on the vector space \mathcal{X}_H . We have a direct sum decomposition $\mathcal{X}_H = \bigoplus_{g\pi} \mathcal{X}_{H,g\pi}$ where $g\pi$ runs over the cosets \bar{Q}/π and $\mathcal{X}_{H,g\pi} = \{f \in \mathcal{X}_H; \text{supp}(f) \in g\pi\}$. Let \mathcal{H}_H be the endomorphism algebra of the \bar{Q} -module \mathcal{X}_H . If $f \in \mathcal{X}_H$ we define $f' \in \mathcal{X}$ by $f'(g) = f(\kappa(g))$ if $g \in Q$ and $f'(g) = 0$ if $g \in G(K) - Q$. We regard \mathcal{X}_H as a subspace of \mathcal{X} via $f \mapsto f'$. Let $\mathcal{H}' = \bigoplus_{\Theta} \mathcal{H}_{\Theta} \subset \mathcal{H}$ where Θ runs over the good (P, P) -double cosets in $G(K)$ that are contained in Q . This is a subalgebra of \mathcal{H} which can be identified with \mathcal{H}_H in such a way that the \mathcal{H}' -module structure on \mathcal{X}_H (restriction of the \mathcal{H} -module structure on \mathcal{X}) coincides with the obvious \mathcal{H}_H -module structure on \mathcal{X}_H . Thus \mathcal{H}_H may be identified with a subalgebra of \mathcal{H} (namely \mathcal{H}').

3.5. The finite dimensional vector space V^{U_Q} of U_Q -invariants on V is naturally a \bar{Q} -module (since Q normalizes U_Q). This \bar{Q} -module can be decomposed as $E_1 \oplus E_2 \oplus \dots \oplus E_r$ where E_1, \dots, E_r are irreducible \bar{Q} -modules. We show:

(a) For any $i \in [1, r]$, the space of invariants $E_i^{U_P/U_Q}$ is nonzero and (when regarded as a P/U_P -module in the obvious way) is a direct sum of copies of \mathbf{E} . By a known property of unipotent representations of finite reductive groups (applied to the Q/U_Q -module E_i) it is enough to show that the π/u_π -module $E_i^{u_\pi}$ contains a copy of \mathbf{E} (it is then nonzero and a direct sum of copies of \mathbf{E}).

We can find a parabolic subgroup π' of Q/U_Q and an irreducible cuspidal representation \mathbf{E}_0 of $\pi'/u_{\pi'}$ such that the $\pi'/u_{\pi'}$ -module $E_1^{u_{\pi'}}$ contains \mathbf{E}_0 . Then the $\pi'/u_{\pi'}$ -module $(V^{U_Q})^{u_{\pi'}}$ contains \mathbf{E}_0 . Let P' be the inverse image of π' under $Q \rightarrow Q/U_Q$; then $P' \in \mathcal{P}$ is contained in Q and the $P'/U_{P'}$ -module $(V^{U_Q})^{U_{P'}}$ contains \mathbf{E}_0 . (We have $P'/U_{P'} = \pi'/u_{\pi'}$.) Replacing P' by a Q -conjugate we can assume that $P' = P_{J_1}$ for some $J_1 \subset H$. We have $U_Q \subset U_{P'}$ hence $V^{U_{P'}} \subset V^{U_Q}$ and $(V^{U_Q})^{U_{P'}} = V^{U_{P'}}$. Thus the $P'/U_{P'}$ -module $V^{U_{P'}}$ contains \mathbf{E}_0 . By the uniqueness of cuspidal support of V we see that \mathbf{E}_0 is a unipotent (cuspidal) representation and then using [L4, 1.6(b)] we see that we can find $g' \in G(K)$ such that $\text{Ad}(g')$ carries P to P' and \mathbf{E} to a representation isomorphic to \mathbf{E}_0 . In particular we have $\sharp(J) = \sharp(J_1)$. We also see that the reductive quotients of the parabolic subgroups π and π' of Q/U_Q both admit unipotent cuspidal representations. By the classification of unipotent cuspidal representations we see that these two parabolic subgroups are conjugate under Q/U_Q . Hence P, P' are conjugate under Q and in particular under $G(K)'$. Since $P = P_J, P' = P_{J_1}$ it follows that $P = P'$ and g' above is such that $\text{Ad}(g')$ carries P to $P' = P$ and \mathbf{E} to a representation isomorphic to \mathbf{E}_0 . By [L4, 1.16], $\text{Ad}(g')$ carries \mathbf{E} to a representation isomorphic to

E. Thus \mathbf{E}_0 is isomorphic to \mathbf{E} . We see that the P/U_P -module $E_i^{U_P/U_Q}$ contains a copy of \mathbf{E} . This completes the proof of (a).

Replacing Q by P in (a) we deduce:

(b) The (finite dimensional) P/U_P -module V^{U_P} is \mathbf{E} -isotypic.

3.6. We preserve the setup of 3.4.

Let \mathcal{C}_1 be the category of (finite dimensional) representations \mathcal{V} of Q/U_Q such that the π/u_π -module \mathcal{V}^{u_π} of u_π -invariants is a direct sum of copies of \mathbf{E} and it generates the Q/U_Q -module \mathcal{V} . Note that each irreducible object in \mathcal{C}_1 is a unipotent representation of Q/U_Q which belongs to \mathcal{C}_1 . Let \mathcal{C}_2 be the category of (finite dimensional) representations of the Hecke algebra \mathcal{H}_H . Let $\mathcal{V} \in \mathcal{C}_1$. By Frobenius reciprocity we have an isomorphism $\mathrm{Hom}_{Q/U_Q}(\mathcal{X}_H, \mathcal{V}) \xrightarrow{\sim} \mathrm{Hom}_\pi(\mathbf{E}, \mathcal{V})$ induced by the imbedding $\mathbf{E} \rightarrow \mathcal{X}_H$ which takes $e \in \mathbf{E}$ to the function $f_e : Q/U_Q \rightarrow \mathbf{E}$ given by $h \mapsto h^{-1}e$ if $h \in \pi$ and $g \mapsto 0$ if $g \notin \pi$. Since $\mathrm{Hom}_{Q/U_Q}(\mathcal{X}_H, \mathcal{V})$ is an \mathcal{H}_H -module in an obvious way, we see that $\mathrm{Hom}_\pi(\mathbf{E}, \mathcal{V})$ is naturally an \mathcal{H} -module. Clearly we have $\mathrm{Hom}_\pi(\mathbf{E}, \mathcal{V}) = \mathrm{Hom}_{\pi/u_\pi}(\mathbf{E}, \mathcal{V}^{u_\pi})$. Hence $\mathcal{V}_\mathbf{E} := \mathrm{Hom}_{\pi/u_\pi}(\mathbf{E}, \mathcal{V}^{u_\pi})$ is an object of \mathcal{C}_2 . Note that $\mathcal{V} \mapsto \mathcal{V}_\mathbf{E}$ is an equivalence of categories.

For each $D \in \mathcal{C}_2$ we denote by $[D]$ the object of \mathcal{C}_1 (well defined up to isomorphism) such that $[D]_\mathbf{E} \cong D$ in \mathcal{C}_2 .

By 3.5(a), we have $V^{U_Q} \in \mathcal{C}_1$ and

$$V_\mathbf{E}^{U_Q} = \mathrm{Hom}_{\pi/u_\pi}(\mathbf{E}, V^{U_Q}) = \mathrm{Hom}_{\pi/u_\pi}(\mathbf{E}, (V^{U_Q})^{u_\pi}) = \mathrm{Hom}_P(\mathbf{E}, V^{U_P}) \in \mathcal{C}_2$$

hence

$$V^{U_Q} \cong [\mathrm{Hom}_P(\mathbf{E}, V^{U_P})].$$

From the definitions we see that the \mathcal{H}_H -module $\mathrm{Hom}_P(\mathbf{E}, V^{U_P})$ is the restriction of the \mathcal{H} -module $\mathrm{Hom}_P(\mathbf{E}, V^{U_P})$ (see 3.3) to the subalgebra \mathcal{H}_H . We see that:

(a) *The Q/U_Q -module V^{U_Q} is isomorphic to a direct sum of unipotent representations $[D_1] \oplus [D_2] \oplus \dots \oplus [D_r]$ where D_1, D_2, \dots, D_r are irreducible \mathcal{H}_H -modules such that the \mathcal{H}_H -module $D_1 \oplus D_2 \oplus \dots \oplus D_r$ is isomorphic to the restriction of the \mathcal{H} -module $\mathrm{Hom}_P(\mathbf{E}, V^{U_P})$ to the subalgebra \mathcal{H}_H of \mathcal{H} .*

3.7. Let $P = P_J$, \mathbf{E} be as in 3.2 and let (V, σ) be as in 3.3. Let $H \subsetneq \mathbf{I}$, $Q = P_H$, $\underline{Q} = \underline{P}_H$ be as in 3.4. Let \mathcal{E} be the \mathcal{H} -module $\mathrm{Hom}_P(\mathbf{E}, V^{U_P})$. Let $\gamma \in G(K)_{vrs}$ be such that $\gamma \in Q = P_H$. Let $T' = T'_\gamma$. Let \bar{g} be the image of γ in Q/U_Q . Note that \bar{g} is an F_q -rational regular semisimple element of $\underline{Q}/\underline{U}_\underline{Q}$, see 2.4(b). Let \mathbf{c} be the conjugacy class in W_H defined as follows: if β' is a Borel subgroup of $\underline{Q}/\underline{U}_\underline{Q}$ containing \bar{g} and β'' is the image of β' under the Frobenius map then β', β'' are in relative position which belongs to \mathbf{c} . Note that $\mathbf{c} \subset C_{T'}$ where $C_{T'}$ is the conjugacy class in W' associated to T' in 2.5(a). For any $w \in \mathbf{c}$ let R_w be the virtual representation of the finite reductive group Q/U_Q associated to w as in [DL]. Since R_w is independent of w as long as $w \in \mathbf{c}$, we denote it by $R_\mathbf{c}^H$. According to [DL, 7.9] for any unipotent representation E of Q/U_Q we have

$$\mathrm{tr}(\bar{g}, E) = (E : R_{\mathbf{c}}^H)$$

where $(:)$ denotes multiplicity. Now 3.6 implies

$$(a) \quad \mathrm{tr}(\gamma, V^{U_Q}) = \sum_{i \in [1, r]} \mathrm{tr}(\bar{g}, [D_i]) = \sum_{i \in [1, r]} ([D_i] : R_{\mathbf{c}}^H) \in \mathbf{Z}.$$

3.8. For V, P as above and for $Q = P_H$ with $H \subsetneq \mathbf{I}$ (but without assuming that $P \subset Q$) we show:

(a) If $V^{U_Q} \neq 0$ then $J \subset \omega(H)$ for some $\omega \in \Omega$.

(Note that Ω (see 2.1) acts by conjugation on $\{s_i; i \in \mathbf{I}\}$ and this induces an action of Ω on \mathbf{I} .) The proof is almost a repetition of that of 3.5(a).

We can find a parabolic subgroup π' of Q/U_Q and an irreducible cuspidal representation \mathbf{E}_0 of $\pi'/u_{\pi'}$ such that the $\pi'/u_{\pi'}$ -module $E_1^{u_{\pi'}}$ contains \mathbf{E}_0 . Then the $\pi'/u_{\pi'}$ -module $(V^{U_Q})^{u_{\pi'}}$ contains \mathbf{E}_0 . Let P' be the inverse image of π' under $Q \rightarrow Q/U_Q$; then $P' \in \mathcal{P}$ is contained in Q and the $P'/U_{P'}$ -module $(V^{U_Q})^{U_{P'}}$ contains \mathbf{E}_0 . (We have $P'/U_{P'} = \pi'/u_{\pi'}$.) Replacing P' by a Q -conjugate we can assume that $P' = P_{J_1}$ for some $J_1 \subset H$. We have $U_Q \subset U_{P'}$ hence $V^{U_{P'}} \subset V^{U_Q}$ and $(V^{U_Q})^{U_{P'}} = V^{U_{P'}}$. Thus the $P'/U_{P'}$ -module $V^{U_{P'}}$ contains \mathbf{E}_0 . By the uniqueness of cuspidal support of V we see that \mathbf{E}_0 is a unipotent (cuspidal) representation and then using [L4, 1.6(b)] we see that we can find $g' \in G(K)$ such that $\mathrm{Ad}(g')$ carries P to P' and \mathbf{E} to a representation isomorphic to \mathbf{E}_0 . In particular we have $\mathrm{Ad}(g')(P) \subset Q$, proving (a).

4. THE CHARACTER FORMULA

4.1. Let T be as in 1.3 and let B be as in 3.1. Let $J \subsetneq \mathbf{I}$, $P = P_J$, \mathbf{E} be as in 3.2 and let (V, σ) be as in 3.3. Let ϕ_V be as in 1.1. In this section we shall assume that

(a) K has characteristic zero and that p is sufficiently large, so that we are able to quote results of [AK].

Let $\gamma \in G(K)_{\mathrm{cvt}}$. We want to compute $\phi_V(\gamma)$. As in 2.7(b) we associate to γ a subset $H \subsetneq \mathbf{I}$ so that (after replacing if necessary γ by a $G(K)'$ -conjugate), $Q := P_H$ contains γ and (setting $T' = T'_\gamma$) we have $T'_{*s} \subset T_*$ (notation of 2.7). Let M be the centralizer of T'_{*s} , see 2.7; this is the Levi subgroup of a parabolic subgroup of G . Let T'_0 (resp. T'_1) be the maximal compact subgroup (resp. the maximal pro- p subgroup) of $T'(K)$. Let $Q = Q_0, Q_1, Q_2, \dots$ be the strictly decreasing Moy-Prasad filtration [MP] of Q . Then $Q_1 = U_Q$. By [AK] (which is applicable in view of (a)),

(b) ϕ_V is constant on $\cup_{x \in G(K)} x\gamma T'_1 x^{-1}$.

Proposition 4.2. $\gamma Q_1 \subset \cup_{x \in G(K)} x\gamma T'_1 x^{-1}$.

For $n \geq 0$ we set $\mathcal{M}_n := M \cap Q_n$, $T'_n = T' \cap Q_n$. (This agrees with the earlier definition of T'_0, T'_1 .) Note that $Q_0/Q_1 \simeq \mathcal{M}_0/\mathcal{M}_1$ and Q_i/Q_{i+1} is abelian for $i \geq 1$. Moreover, $\mathcal{M}_0/\mathcal{M}_1$ (resp. T'_0/T'_1) is the group of the F_q -points of a reductive group over F_q whose Lie algebra is denoted by \mathfrak{m} (resp. \mathfrak{t}').

We need the following two lemmas:

Lemma 4.3. *Let $n \geq 1$ and $\gamma' \in \gamma T'_1$. Then, for any $z_n \in \mathcal{M}_n$, there exist $g_n \in \mathcal{M}_n$, $\gamma_n \in T'_n$ such that $g_n \gamma' z_n g_n^{-1} \in \gamma' \gamma_n \mathcal{M}_{n+1}$.*

Proof. If $\mathcal{M}_n = \mathcal{M}_{n+1}$, one can take $z_n = 1 = g_n$. If $\mathcal{M}_n \neq \mathcal{M}_{n+1}$, there is a natural isomorphism $\iota_n : \mathcal{M}_n / \mathcal{M}_{n+1} \rightarrow \mathfrak{t}'(F_q) \oplus (\mathfrak{m}/\mathfrak{t}')(F_q)$. Note that since $\gamma' \in G(K)_{cvt}$, $d := \text{Ad}(\gamma'^{-1}) - 1$ induces an isomorphism $(\mathfrak{m}/\mathfrak{t}')(F_q) \rightarrow (\mathfrak{m}/\mathfrak{t}')(F_q)$. Let $z_n \in \mathcal{M}_n$ and $\iota_n(z_n) = t_n + m_n$ with $t_n \in \mathfrak{t}'(F_q)$ and $m_n \in (\mathfrak{m}/\mathfrak{t}')(F_q)$. Choose $\gamma_n \in T'_n$ and $g_n \in \mathcal{M}_n$ so that $\iota_n(\gamma_n) = t_n$ and $\iota_n(d(g_n)) = -m_n$. Then $\gamma'^{-1} g_n \gamma' z_n g_n^{-1} \in \gamma_n \mathcal{M}_{n+1}$. Setting $z_{n+1} = \gamma_n^{-1} \gamma'^{-1} g_n \gamma' z_n g_n^{-1}$, we have $g_n \gamma' z_n g_n^{-1} = \gamma' \gamma_n z_{n+1} \in \gamma' \gamma_n \mathcal{M}_{n+1}$. \square

Lemma 4.4. *Let $n \geq 1$. For any $\gamma' \in \gamma T'_1$ and $z \in Q_n$, there exist $g \in Q_n$, $\gamma_n \in T'_n$ and $z' \in Q_{n+1}$ such that $\text{Ad}(g)(\gamma' z) = \gamma' \gamma_n z'$.*

Proof. Let $Z = \{\alpha \in R; U_\alpha \cap Q_n \supsetneq U_\alpha \cap Q_{n+1}\}$. If $Z = \emptyset$, then $Q_n = \mathcal{M}_n Q_{n+1}$. Hence $z = z_n z'$ for some $z_n \in \mathcal{M}_n$, $z' \in Q_{n+1}$ and one can take $g = g_n$ and γ_n as in Lemma 4.3. If $Z \neq \emptyset$, then $\mathcal{M}_n = \mathcal{M}_{n+1}$ and $T'_n = T'_{n+1}$. Moreover, since $\gamma' \in G(K)_{cvt}$, $d := \text{Ad}(\gamma'^{-1}) - 1$ induces an isomorphism $Q_n / Q_{n+1} \rightarrow Q_n / Q_{n+1}$. Let $g \in Q_n$ be such that $d(g) = z^{-1}$. Then, $\gamma'^{-1} g \gamma' z g^{-1} \in Q_{n+1}$. Setting $z' = \gamma'^{-1} g \gamma' z g^{-1}$ and $\gamma_n = 1$, we have $g \gamma' z g^{-1} = \gamma' \gamma_n z' \in \gamma' \gamma_n Q_{n+1}$. \square

Continuing with the proof of Proposition 4.2, let $\gamma_0 = \gamma$ and $z_1 \in Q_1$. We will construct inductively sequences g_1, g_2, \dots , t_1, t_2, \dots and z_1, z_2, \dots such that $g_i \in Q_i$, $\gamma_i \in T'_i$, $z_i \in Q_i$ and $\text{Ad}(g_k \cdots g_2 g_1)(\gamma_0 z_1) = \text{Ad}(g_k)(\gamma_0 \gamma_1 \cdots \gamma_{k-1} z_k) = (\gamma_0 \gamma_1 \cdots \gamma_k) z_{k+1}$.

Applying Lemma 4.4 with $n = 1$, $\gamma' = \gamma_0$ and $z = z_1$, we find $\gamma_1 \in T'_1$ and $z_2 \in Q_2$ such that $g_1 \gamma_0 z_1 g_1^{-1} = \gamma_0 \gamma_1 z_2$ with $\gamma_1 \in T'_1$ and $z_2 \in Q_2$. Suppose that we have found $g_i \in Q_i$, $z_{i+1} \in Q_{i+1}$ and $\gamma_i \in T'_i$ for $i = 1, \dots, k$ where $k \geq 1$. Applying Lemma 4.4 with $n = k + 1$, $\gamma' = \gamma_0 \gamma_1 \cdots \gamma_k$ and $z = z_{k+1}$, we find $g_{k+1} \in Q_{k+1}$, $\gamma_{k+1} \in T'_{k+1}$ and $z_{k+2} \in Q_{k+2}$ such that

$$g_{k+1} \gamma_0 \gamma_1 \cdots \gamma_k z_{k+1} g_{k+1}^{-1} = \text{Ad}(g_{k+1} \cdots g_2 g_1)(\gamma_0 z_1) = \gamma_0 \gamma_1 \gamma_2 \cdots \gamma_{k+1} z_{k+2}.$$

Taking $g \in Q_1$ to be the limit of $g_k \cdots g_2 g_1$ as $k \rightarrow \infty$, we have $\text{Ad}(g)(\gamma z_1) \in \gamma T'_1$. \square

Note that γ acts on $V^{Q_1} = V^{U_Q}$ since γ normalizes Q_1 .

Proposition 4.5. $\phi_V(\gamma) = \text{tr}(\gamma; V^{U_Q})$.

Proof. If $f : G(K) \rightarrow \mathbf{C}$ is any locally constant function with compact support then there is a well defined linear map $\sigma_f : V \rightarrow V$ such that for any $x \in V$ we have $\sigma_f(x) = \int_G f(g) \sigma(g)(x) dg$. (dg is the Haar measure on $G(K)$ for which $\text{vol}(B) = 1$). This linear map has finite rank hence it has a well defined trace $\text{tr}(\sigma_f) \in \mathbf{C}$.

Note that $Q_1 \gamma Q_1 = \gamma Q_1$. Let f_γ be the characteristic function of γQ_1 . Then,

by 4.1(b) and Proposition 4.2 we have

$$\begin{aligned} \mathrm{tr}(\sigma_{f_\gamma}) &= \int_G f_\gamma(g) \phi_V(g) dg \\ &= \int_{Q_1 \gamma Q_1} \phi_V(\gamma) dg = \mathrm{vol}(Q_1 \gamma Q_1) \phi_V(\gamma) = \mathrm{vol}(Q_1) \phi_V(\gamma). \end{aligned}$$

Moreover, σ_{f_γ} maps V onto V^{Q_1} , and on V^{Q_1} , it is equal to $\sigma(\gamma)$ times $\mathrm{vol}(Q_1)$. Hence $\mathrm{tr}(\sigma_{f_\gamma}) = \mathrm{tr}(\sigma_{f_\gamma}; V^{Q_1}) = \mathrm{vol}(Q_1) \mathrm{tr}(\sigma(\gamma)|V^{Q_1})$ and the proposition follows. \square

Note that the proposition above is in fact valid (with the same proof) for any irreducible admissible (not necessarily unipotent) representation V of depth 0 of $G(K)$. We now state the main result of this paper.

Theorem 4.6. *We preserve the setup of 4.1.*

- (a) *If $J \not\subset \omega(H)$ for any $\omega \in \Omega$ then $\phi_V(\gamma) = 0$.*
- (b) *Assume that $J \subset H$. Then*

$$\phi_V(\gamma) = \sum_{i \in [1, r]} ([D_i] : R_{\mathbf{c}}^H) \in \mathbf{Z}$$

(notation of 3.7).

We prove (a). Assume that $\phi_V(\gamma) \neq 0$. By 4.5 we then have $V^{U_Q} \neq 0$ hence by 3.8(a), $P_J \subset P_{\omega(H)}$ for some $\omega \in \Omega$. It follows that $J \subset \omega(H)$ and (a) is proved. Now (b) is obtained by combining 4.5 with 3.7(a).

We expect that the Theorem holds without the assumption 4.1(a).

4.7. The case where $J \subset \omega(H)$ for some $\omega \in \Omega$ can be reduced to the case 4.6(b) as follows. Let $\tilde{\omega}$ be an element in the normalizer of B in $G(K)$ that represents ω ; we can assume that $\tilde{\omega}$ normalizes T . Let $\gamma' = \tilde{\omega}^{-1} \gamma \tilde{\omega}$, $\tilde{Q} = \tilde{\omega}^{-1} Q \tilde{\omega}^{-1} = P_{\omega^{-1}(H)}$. Now 4.6(b) is applicable to γ' , \tilde{Q} instead of γ , Q and it yields a formula for $\phi_V(\gamma')$. Since γ, γ' are conjugate in $G(K)$ we have $\phi_V(\gamma) = \phi_V(\gamma')$ so that we get a formula for $\phi_V(\gamma)$.

4.8. Let C be a conjugacy class of elements of finite order in W' and let γ', γ'' be two elements in $G(K)_{\mathrm{cvt}}$ such that (denoting by T', T'' the maximal tori in G that contain γ', γ'') we have $C_{T'} = C_{T''} = C$. We show that

- (a) $\phi_V(\gamma') = \phi_V(\gamma'')$.

(It follows that $\phi_V(\gamma)$ is independent of γ as long as $\gamma \in G(K)_{\mathrm{cvt}}$ gives rise to a fixed conjugacy class in W' .)

We can assume that $\gamma' = \gamma$ and $Q = P_H$ are as in 4.1. We can also assume that $T'' = T'$ so that $\gamma'' \in T'$. Using 2.8(a) we see that Q is a minimal parahoric subgroup of $G(K)$ containing γ'' . Using 4.6 and 4.7 we now see that (a) holds.

5. EXAMPLES AND COMMENTS

5.1. In this and the next two subsections we assume that G is adjoint and that 4.1(a) holds. We denote by G^* a semisimple group over \mathbf{C} of type dual to that of G . The unipotent representations of $G(K)$ were classified in [L4] (extending the classification in [KaL] of Iwahori-spherical representations). Namely in [L4, 5.21] a bijection is established between the set \mathcal{U} of unipotent representations of $G(K)$ (up to isomorphism) and the set \mathfrak{S} consisting of all triples (s, u, ρ) (modulo the natural action of G^*) where s is a semisimple element of G^* , u is a unipotent element of G^* such that $sus^{-1} = u^q$ and ρ is an irreducible representation (up to isomorphism) of the group $\bar{Z}(s, u)$ of components of the centralizer $Z(s, u)$ of s and u in G^* such that the centre of G^* acts trivially on ρ ; let $V_{s,u,\rho}$ be the unipotent representation corresponding to (s, u, ρ) .

In the case where $V = V_{s,u,\rho}$ in Theorem 4.6 is tempered, the definition of V in terms of equivariant homology given in [L4] shows that the restriction of the \mathcal{H} -module $\text{Hom}_P(\mathbf{E}, V^{U_P})$ (see 3.3) to the subalgebra \mathcal{H}_H is explicitly computable (in terms of generalized Green functions); moreover, the multiplicities $([D_i] : R_{\mathbf{c}}^H)$ in 4.6(b) are also explicitly known from [L2, 4.23] (in terms of the nonabelian Fourier transform of [L2, 4.14]). We see that in this case the character values $\phi_V(\gamma)$ in 4.6 are explicitly computable integers.

5.2. We have a partition $\mathfrak{S} = \sqcup_{\mathbf{c}} \mathfrak{S}_{\mathbf{c}}$ where \mathbf{c} runs over the unipotent conjugacy classes in G^* and $\mathfrak{S}_{\mathbf{c}}$ is the set of (equivalence classes of) triples (s, u, ρ) in \mathfrak{S} with $u \in \mathbf{c}$. Let $\mathcal{U}_{\mathbf{c}}$ be the subset of \mathcal{U} corresponding to $\mathfrak{S}_{\mathbf{c}}$ under the bijection $\mathcal{U} \leftrightarrow \mathfrak{S}$ in 5.1.

Now assume that \mathbf{c} is a distinguished unipotent class in G^* and let $u \in \mathbf{c}$.

Let $\xi : SL_2(\mathbf{C}) \rightarrow G^*$ be a homomorphism of algebraic groups which carries $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to u . Let Γ be the centralizer in G^* of the image of ξ . Note that Γ is a finite group. Let $M(\Gamma)$ be the set of all pairs (y, r) where y is an element of Γ defined up to conjugacy and r is an irreducible representation of the centralizer $Z_{\Gamma}(y)$ of y in Γ , up to isomorphism. Let $M_1(\Gamma)$ be the set of all $(y, r) \in M(\Gamma)$ such that the centre of G^* acts trivially on r . The equivalence class in $\mathfrak{S}_{\mathbf{c}}$ of a triple (s, u, ρ) gives rise to an element $(y, r) \in M_1(\Gamma)$ as follows. By [KaL, 2.4] we can assume that $s\xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} s^{-1} = \xi \begin{pmatrix} a & qb \\ q^{-1}c & d \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{C})$. Then $\xi \begin{pmatrix} \sqrt{q}^{-1} & 0 \\ 0 & \sqrt{q} \end{pmatrix} s$ is in Γ and is denoted by y . Note that $\bar{Z}(u, s)$ and $Z_{\Gamma}(y)$ can be naturally identified so that ρ can be viewed as an irreducible representation r of $Z_{\Gamma}(y)$. This gives a map from $\mathfrak{S}_{\mathbf{c}}$ to $M_1(\Gamma)$ which is in fact a bijection. We shall write $V_{\mathbf{c};y,r}$ instead of $V_{s,u,\rho}$ when (y, r) corresponds as above to $(s, u, \rho) \in \mathfrak{S}_{\mathbf{c}}$. Let $M'_1(\Gamma)$ be the set of all $(y, r) \in M_1(\Gamma)$ such that $V = V_{\mathbf{c};y,r}$ has a nonzero space of B -invariants V^B . For such (y, r) , V^B is a square integrable irreducible representation of the ordinary affine Hecke algebra \mathcal{H} . By the K -theoretic construction of this representation [KaL], V^B is the specialization at $v = \sqrt{q}$ of a representation of the affine Hecke algebra with parameter v ; this representation can be specialized at $v = 1$, yielding a (finite

dimensional) representation $A_{y,r}$ of W (which in particular is a representation of W'). For $(y, r), (y', r')$ in $M(\Gamma)$ let $\{(y, r), (y', r')\} \in \mathbf{C}$ be the $((y, r), (y', r'))$ -entry of the nonabelian Fourier transform matrix on $M(\Gamma)$ (see [L2, 4.14]).

It seems likely that in the case where $V \in \mathcal{U}_{\mathfrak{c}}$, the character formula in 4.6 can be rewritten in the following form (we write $V = V_{\mathfrak{c}; y, r}$ where $(y, r) \in M_1(\Gamma)$):

$$(a) \quad \phi_V = \sum_{(y', r') \in M'_1(\Gamma)} \{(y, r), (y', r')\} \tau_{A_{y', r'}}$$

as functions $G(K)_{cvt} \rightarrow \mathbf{Z}$, where $\tau_{A_{y', r'}}$ is as in 2.3.

For example, (a) holds when G is of type E_8 and \mathfrak{c} is the regular, or subregular or subsubregular unipotent class in G^* . (In these cases we have $\Gamma = \{1\}$.)

5.2. In this subsection we assume that G is of type G_2 and \mathfrak{c} is the subregular unipotent class in G^* . In this case we have $\Gamma = S_3$ and $\mathcal{U}_{\mathfrak{c}}$ consists of eight square integrable irreducible representations of $G(K)$ considered in [L1, 1.7]. Four representations in $\mathcal{U}_{\mathfrak{c}}$ (denoted by V, V', V'', V''') are Iwahori-spherical and the four corresponding irreducible representations of the affine Hecke algebra of type G_2 are carried by the W -graph denoted in [L1, 3.13, Type \tilde{G}_2] by $\mathcal{G}, \mathcal{G}', \mathcal{G}'', \mathcal{G}'''$ (they have dimension 3, 3, 2, 1 and \mathcal{G} gives rise to the reflection representation of the affine Hecke algebra). The other four representations in $\mathcal{U}_{\mathfrak{c}}$ are supercuspidal and are denoted by S, S', S'', S''' . The W -graphs $\mathcal{G}, \mathcal{G}', \mathcal{G}'', \mathcal{G}'''$ also carry (for $q = 1$) representations of W of dimension 3, 3, 2, 1 denoted by A, A', A'', A''' . From Theorem 4.6 we can deduce:

$$\phi_V = (1/6)\tau_A + (1/2)\tau_{A'} + (1/3)\tau_{A''} + (1/3)\tau_{A'''},$$

$$\phi_{V'} = (1/2)\tau_A + (1/2)\tau_{A'},$$

$$\phi_{V''} = (1/3)\tau_A + (2/3)\tau_{A''} - (1/3)\tau_{A'''},$$

$$\phi_{V'''} = (1/3)\tau_A - (1/3)\tau_{A''} + (2/3)\tau_{A'''},$$

$$\phi_S = (1/6)\tau_A - (1/2)\tau_{A'} + (1/3)\tau_{A''} + (1/3)\tau_{A'''},$$

$$\phi_{S'} = (1/2)\tau_A - (1/2)\tau_{A'},$$

$$\phi_{S''} = \phi_{S'''} = (1/3)\tau_A - (1/3)\tau_{A''} - (1/3)\tau_{A'''},$$

as functions $G(K)_{cvt} \rightarrow \mathbf{Z}$. We see that in our case the equality 5.1(a) holds.

5.3. The functions τ_A for $A = A_{y,r}$ ($(y, r) \in M'_1(\Gamma)$ in the setup of 5.1) can be regarded as p -adic analogues of the (uniform) almost characters (see [L2]) for the corresponding reductive group over F_q (restricted to F_q -rational regular semisimple elements). It would be interesting to see whether these functions have natural extensions to $G_{rs} \cap G(K)$ (in the same way as the uniform almost characters of reductive groups over F_q are defined on the whole group not just on regular

semisimple elements) and then to see whether these extensions play a role in computing ϕ_V on $G_{rs} \cap G(K)$. One possible way to find these extensions is to note that, as shown in 2.3, a function of the form τ_A can be viewed as the characteristic function of a "local system" \mathcal{L}_A on $G(\underline{K})_{crr}$. One can hope that each \mathcal{L}_A extends to the set $G_{rs} \cap G(\underline{K})$ as some kind of intersection cohomology complex whose characteristic function will define the desired extension of each τ_A . This would be entirely analogous to what happens for the uniform almost characters, see [L3].

5.4. In the case where $g \in G(K)_{vr}$, the problem of computing the character of a unipotent representation V of $G(K)$ at g can in principle be reduced to the special case (treated in §4) where g is compact very regular. Indeed, the results of [C] reduce this problem to the computation of the character of a Jacquet module of V (which is an admissible representation of finite length of a possibly smaller group with composition factors being unipotent representations computable from [BK, Cor. 8.4]) on a compact very regular element, for which the results in §4 are applicable. (Compare with Theorem 4.3 in [KL].)

5.5. Let us drop the assumption in 1.1 that G is split over K (but we still assume that G is split over \underline{K}). Then the unipotent representations of $G(K)$ are well defined, see [L4] and $G(K)_{crr}$ is defined as in 1.2. It is likely that the character formula 4.6 for unipotent representations of $G(K)$ extends to this more general case with essentially the same proof.

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